Use of Polynomial Chaos Expansion for the modeling of uncertain dynamical systems

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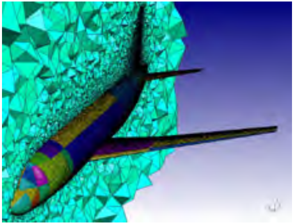
The importance of Modeling

Models are **abstract** representations of the real world



Simulation models are extensively used in civil engineering practice. Such models allow the user to

- understand structural system performance,
- predict structural behavior,
- diagnose damage,
- optimize design, etc

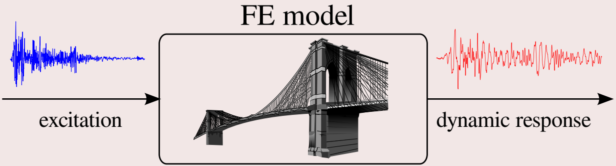
The importance of Modeling

Structures may fail because:

- the models used for their design do not properly represent the complexity of the physics;
- the values of the input parameters have not been selected properly.

Taking **uncertainty** into account in the analysis and design of structures is of crucial importance

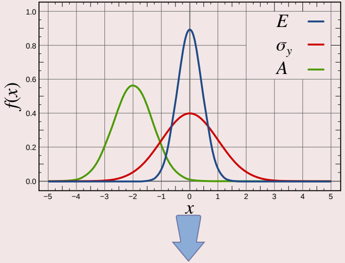
Yet, in a lot of cases a fully detailed simulation is either **unrealizable** or too **costly**.



Simulation of **dynamic response** through FE models requires **excessive computational resources** particularly for complex, large structures.

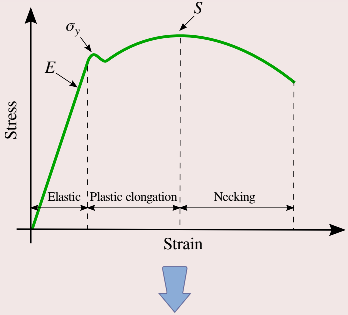
Dynamic response of structural systems with uncertainties subjected to extreme loading conditions.

Structural system is characterized by parameter uncertainty



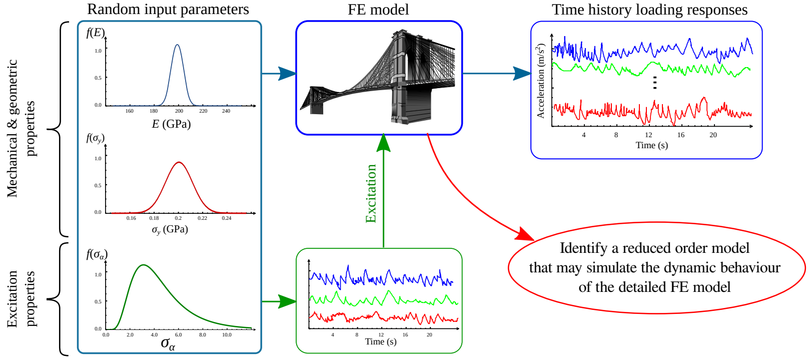
The behavior of the modeled structure has to be examined for a range of structural characteristics.

Nonlinearities are taken into account



The impact of different types of excitation (of different magnitude and/or spectral content) should also be examined.

The Metamodeling Problem



The Metamodeling Problem

Problem definition

Consider a structural system represented by a numerical model $\mathscr{M}$ characterized by uncertain input parameters $\xi = [\xi_1, \xi_2, \ldots, \xi_M]^{\mathsf{T}}$ with known joint pdf $f(\xi)$. The dynamic response of $\mathscr{M}$ to a given input excitation $x[t, \xi]$ will also be a random variable:

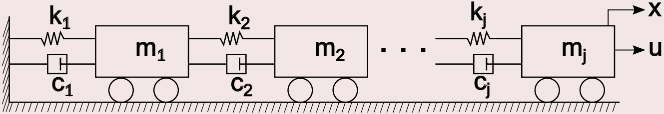
$$y[t, \xi] = \mathscr{M}(x[t, \xi], \xi), \quad t = 1, 2, \ldots, T$$

A metamodel $\tilde{\mathscr{M}}$ which must be able to predict and/or simulate the detailed numerical model results in a computationally inexpensive way and with sufficient accuracy is sought.

Objectives of the study

- Development of a metamodeling method based on PC-NARX models
- Introduction of PC-NARX identification methods for both prediction and simulation purposes.
- The validation of PC-NARX metamodeling method through its application to the case of a five-storey shear frame model subjected to dynamic excitation leading to nonlinear response.

AutoRegressive with eXogenous input (ARX) models



A j-DOF deterministic system is described by the general difference equation

$$x_i = a_1 x_{i-1} + a_2 x_{i-2} + \ldots a_{2j} x_{i-2j} + b_1 u_{i-1} + b_2 u_{i-2} + \ldots + b_{2j-1} u_{i-2j+1}$$

An ARX(n_a, n_b, n_d) is defined as follows:

$$x[t] + \underbrace{\sum_{i=1}^{n_a} a_i \cdot x[t-i]}_{\text{AR part}} = \underbrace{\sum_{i=n_d}^{n_b} b_i \cdot u[t-i]}_{\text{X part}} + w[t], \quad w[t] \sim NID(0, \sigma_w^2)$$

n_a	:	AR order	a_i	:	AR coefficients
n_b	:	X order	b_i	:	X coefficients
n_d	:	delay	$w[t]$	:	model residual sequence

The Metamodeling Problem

Polynomial Chaos Nonlinear ARX (PC-NARX) models

$$y[t] = \sum_{i=1}^{n_\theta} \theta_i(\xi) \cdot g_i(z[t]) + e[t]$$

random parameters $\theta_i(\xi)$ describe the uncertainty propagation. They may be expanded on a PC basis orthogonal to the pdf of the random input variables ξ

$$\theta_i(\xi) = \sum_{j=1}^{p} \theta_{i,j} \cdot \phi_{d(j)}(\xi)$$

$g_i(\cdot)$: nonlinear function operators that reflect the nonlinear structural dynamics

$z[t] = \left[y[t-1], \ldots, y[t-n_a], x[t], \ldots, x[t-n_b]\right]^T$: regression vector
n_a, n_b: maximum output and input time lags
$e[t]$: residual sequence
$\theta_{i,j}$: unknown deterministic coefficients of projection
$d(j)$: multi-indices of the multivariate polynomial basis

n_θ: number of nonlinear regression terms
σ_e^2: residual sequence variance
$\phi_{d(j)}$: basis functions orthonormal w.r.t. the joint pdf of ξ

PC-NARX parameter estimation

- coefficients of projection $\theta_{i,j}$

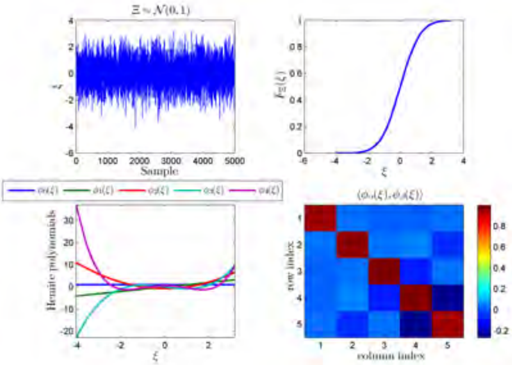
PC-NARX structure selection

- select nonlinear functions $g_i(z[t])$ (polynomial, wavelet, radial basis functions, and so on)
- select PC functional subspace

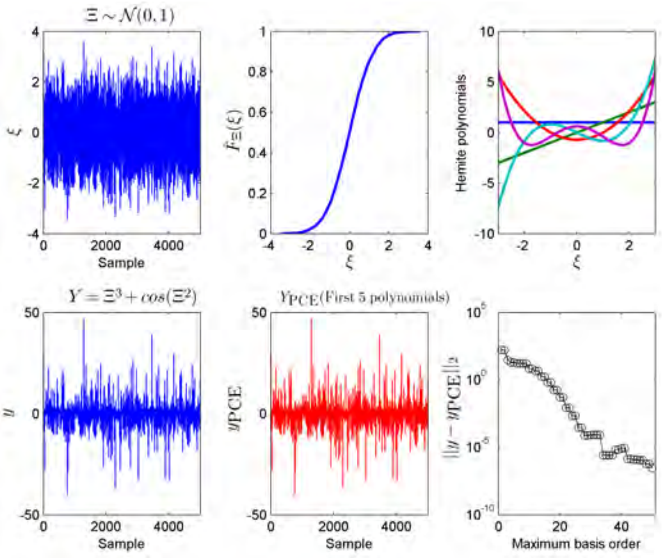
The PC basis basis $\phi_{\mathbf{d}(j)}$ is formed from polynomial that are orthonormal with respect to the joint probability density function of $\boldsymbol{\xi}$. Assume the univariate case (single variable):

$$E[\phi_{\boldsymbol{\alpha}}(\boldsymbol{\xi}), \phi_{\boldsymbol{\beta}}(\boldsymbol{\xi})] = \delta_{\boldsymbol{\alpha},\boldsymbol{\beta}} = \begin{cases} 1 & \text{for } \alpha = \beta \\ 0 & \text{otherwise} \end{cases}$$

PDF	**Support**	**Polynomials**
Normal (Gaussian)	$(-\infty, \infty)$	Hermite
Uniform	$[-1, 1]$	Legendre
Gamma	(0,1)	Laguerre
Chebyshev	(-1, 1)	Chebyshev
Beta	(-1, 1)	Jacobi



Illustrative Example



Estimation of a PC-NARX model for purposes of prediction

Consider K simulations conducted with

$\xi_k = [\xi_{k,1}, \xi_{k,2}, \ldots, \xi_{k,M}]^{\mathsf{T}}$: random input parameter vector realizations

$x_k^T = \{x_k[1,\xi_k], x_k[2,\xi_k], \ldots, x_k[T,\xi_k]\}$: set of input excitation signals ($k = 1, 2, \ldots, K$)

⇓

$y_k^T = \{y_k[1,\xi_k], y_k[2,\xi_k], \ldots, y_k[T,\xi_k]\}$

corresponding set of the full scale numerical model dynamic responses

(assumed to also follow a PC-NARX model)

⇓

Estimation of the coefficients of projection $\theta = [\,\theta_{1,1}, \ldots, \theta_{n_\theta,p}]^{\mathsf{T}}$ based on the minimization of the Prediction Error criterion:

$$\widehat{\theta} = \arg\min_{\theta} \left\{ \sum_{k=1}^{K} \sum_{t=1}^{T} (y_k[t] - \hat{y}_k[t|t-1])^2 \right\} = \arg\min_{\theta} \left\{ \sum_{k=1}^{K} \sum_{t=1}^{T} e_k^2[t] \right\}$$

$\hat{y}_k[t|t-1]$: PC-ARX model's one-step-ahead prediction

⇓

Ordinary Least Squares (OLS) estimator: $\widehat{\theta} = \left(\Phi^{\mathsf{T}}(\xi) \cdot \Phi(\xi)\right)^{-1} \cdot \left(\Phi(\xi)^{\mathsf{T}} \cdot Y\right)$

$\Phi(\xi)$: regression matrix $\quad$ Y : pooled response signal vector

Estimation of a PC-ARX model for purposes of simulation

The simulated response of a given PC-NARX metamodel may be obtained recursively as:

$$\bar{y}_k[t] = \sum_{i=1}^{n_\theta} \theta_i(\xi_k) \cdot g_i(\bar{z}[t]), \quad t = 1, 2, \ldots, T$$

with given initial conditions $\{\bar{y}_k[1-n_a], \ldots, \bar{y}_k[0]\}$ and $\{x_k[1-n_b], \ldots, x_k[0]\}$

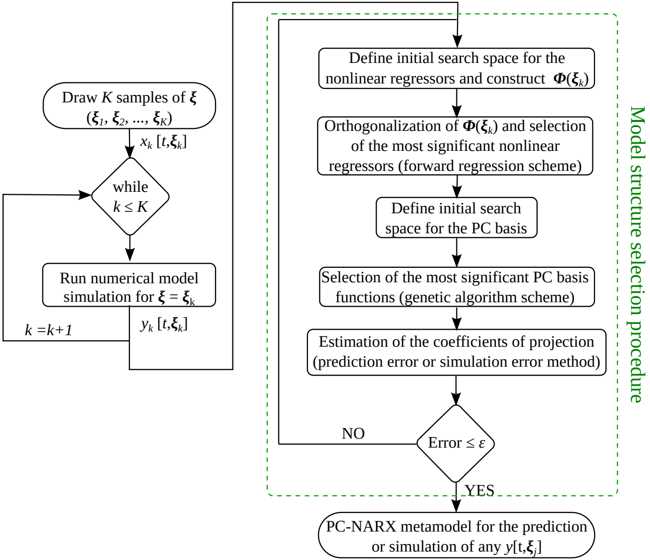
Estimation of the model coefficients of projection θ based on the minimization of the Simulation Error criterion:

$$\widehat{\theta}_s = \arg\min_{\theta_s} \left\{ \sum_{k=1}^{K} \sum_{t=1}^{T} (y_k[t] - \bar{y}_k[t])^2 \right\} = \arg\min_{\theta_s} \left\{ \sum_{k=1}^{K} \sum_{t=1}^{T} \varepsilon_k^2[t] \right\}$$

$\Downarrow$

Iterative nonlinear optimization methods

Flowchart of the complete identification scheme



Estimation of a PC-ARX model for purposes of Simulation

Assuming given initial conditions for the input & output $\{\bar{y}_k[1-n_a],\ldots,\bar{y}_k[0]\}$ and $\{x_k[1-n_b],\ldots,x_k[0]\}$ we may derive the simulated response of a PC-ARX metamodel via the following relationship:

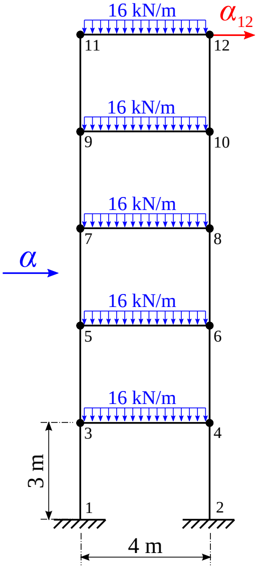
$$\bar{y}_k[t] = -\sum_{i=1}^{n_a} a_i(\boldsymbol{\xi}_k)\cdot\bar{y}_k[t-i] + \sum_{i=0}^{n_b} b_i(\boldsymbol{\xi}_k)\cdot x[t-i], \quad t=1,2,\ldots,T$$

In this case, the estimation of the model coefficients of projection $\boldsymbol{\theta}$ is based on the minimization of the Simulation Error criterion:

$$\widehat{\boldsymbol{\theta}}_s = \arg\min_{\boldsymbol{\theta}_s}\left\{\sum_{k=1}^{K}\sum_{t=1}^{T}(y_k[t]-\bar{y}_k[t])^2\right\} = \arg\min_{\boldsymbol{\theta}_s}\left\{\sum_{k=1}^{K}\sum_{t=1}^{T}\varepsilon_k^2[t]\right\}$$

The relationship is in this case a **nonlinear one** which may be solved by employing **Iterative nonlinear optimization methods** such as the LevenbergMarquardt algorithm (LMA), the Newton-Raphson method or others.

Numerical Application



Simple Implementation Example

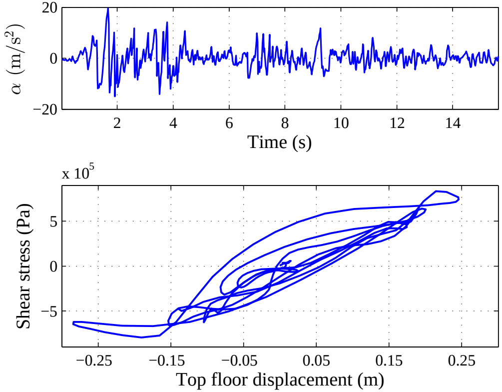
The described framework is implemented for the simulation of the response of a **five-storey shear frame**, subjected to a (known) dynamic input in the form of earthquake excitation.

The frame is described by a nonlinear material law, allowing for the sections to move into the post-yield region which causes nonlinear behavior to occur.

We consider the following **input parameters**:

Input parameter	Vertical elements	Horizontal elements
Density (kg/m^3)	7850	7850
Poisson ratio	0.29	0.29
Young moduli (GPa)	$\mathcal{U}(190, 210)$	$\mathcal{U}(190, 210)$
Yield stress (MPa)	$\mathcal{U}(200, 500)$	$\mathcal{U}(200, 500)$
Cross section area (m^2)	$\mathcal{U}(0.04, 0.09)$	0.0625

One of the recorded acceleration instances for the El Centro earthquake* has been utilized as ground excitation:

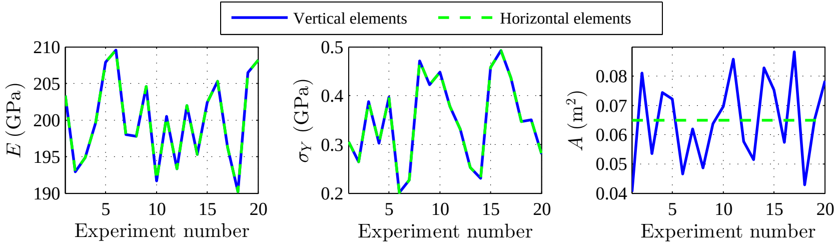


causing the observed shear stress vs top floor displacement response.
The curve shown here corresponds to the first simulation experiment (with $\boldsymbol{\xi}_1$) and $t = 1, 2, \ldots 250$.

* downloadable at: http: //peer.berkeley.edu/peer_ground_motion_database

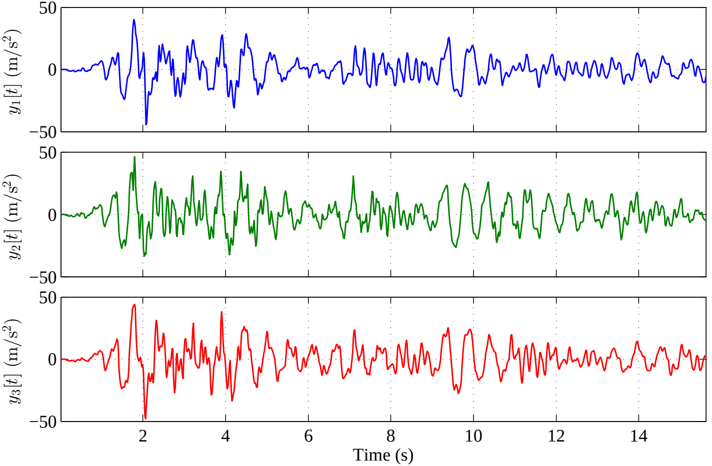
Simulation Experiments

The following visualization illustrates the range of **Material and Geometric properties** of the shear frame model for the 20 simulations conducted using a detailed structural model. The ANSYS finite element software has been used for the reference simulations.



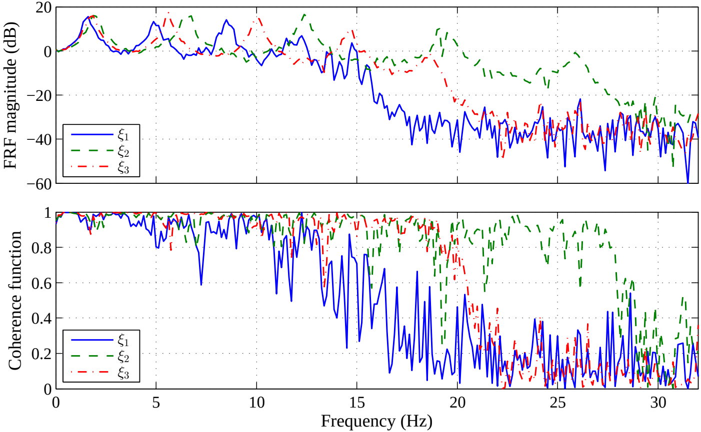
Simulation Experiments

Below the reference numerical model dynamic response signals $y_k[t]$ for separate input parameter vectors $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2$ and $\boldsymbol{\xi}_3$ are plotted.



Simulation Experiments

The magnitude of the estimated FRF (using the Welch method - MATLAB pwelch) and the corresponding estimated coherence function of the dynamic response signals obtained for input parameter vectors $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2$ and $\boldsymbol{\xi}_3$.are plotted



Note:

The **Coherence Function** is a measure used to examine the relation between two signals or data sets. It is expresses the power transfer between input and output of a system.

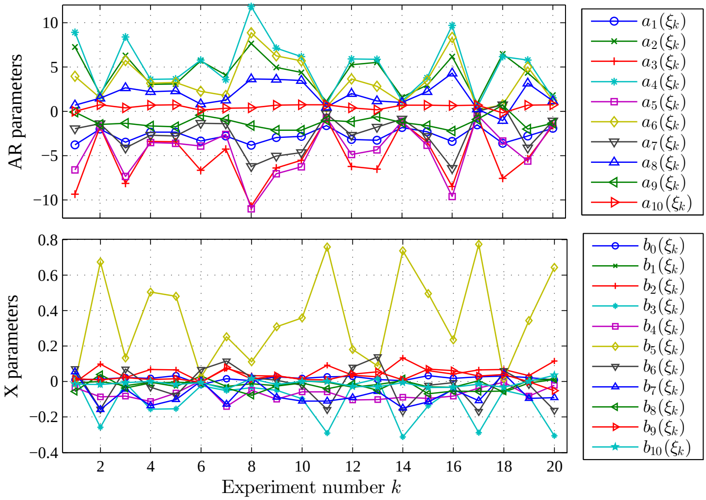
It is defined as:

$$C_{xy} = \frac{|S_{xy}|^2}{S_{xx} S_{yy}}$$

Values of coherence will lie in the range $0 \leq C_{xy} \leq 1$. For an ideal constant parameter linear system with a single input $x(t)$ and single output $y(t)$, the coherence will be equal to one.
In the previous plot, the system corresponding to parameter set $\boldsymbol{\xi}_1$ is therefore the furthest from linearity.

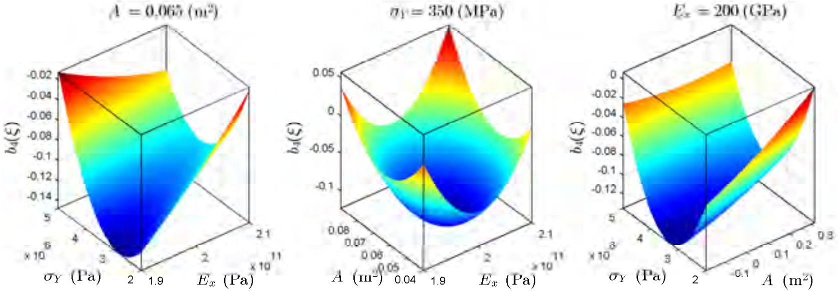
Results

The estimated **PC-ARX model parameters**



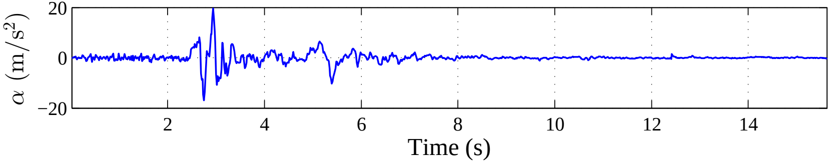
Results

Polynomial expansion of $b_4(\xi)$ model parameter onto the input space



Results

In order to **validate** the workings of the metamodeling framework the performance of the identified PC-ARX(10,10) metamodel is tested for the prediction and simulation of the dynamic response of the FE model subjected this time to the Pacoima Dam earthquake:

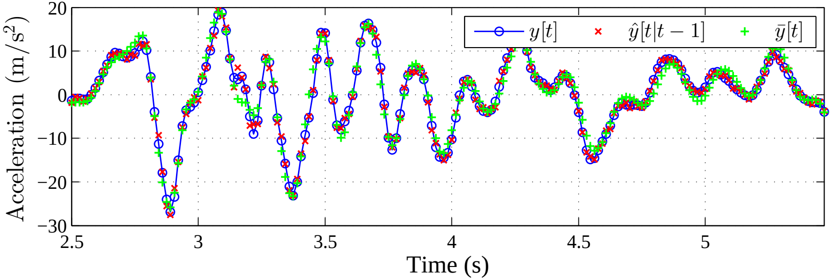


The performance in prediction and simulation is remarkable given the large reduction in computational time.

0.7836 % prediction error
3.7585 % simulation error
5000 times reduced simulation time

Results

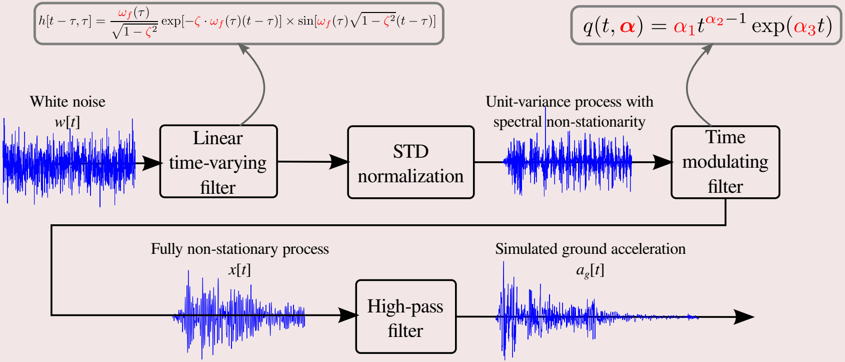
Below the dynamic response of the numerical model and the corresponding PC-ARX(10,10) based one-step-ahead predictions ($\times$) and refined PC-ARX(10,10) based simulations ($+$) ($\boldsymbol{\xi} \neq \boldsymbol{\xi}_k$; $k = 1, \ldots, 20$) are plotted



Summarizing

- Stochastic metamodels of low order that are capable of accurately approximating FE models are developed.
- The metamodeling method is based on NARX models and Polynomial Chaos basis expansion.
- The numerical results demonstrate the efficiency of the proposed methodology for accurate prediction and simulation of the dynamic response of the model.
- The proposed methodology may be adapted as an approximative low cost surrogate for a number of purposes such as vibration control, SHM, model updating and others.

Parametric modelling of the Input - Earthquake Accelerograms
[S. Rezaeian & A.D. Kiureghian 2010]

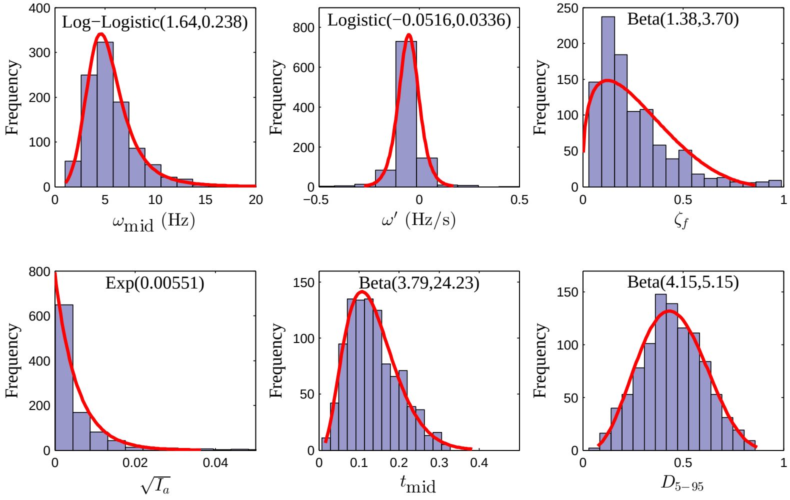


ζ_f: damping ratio of the filter
$\omega_f(\tau) = \omega_{\text{mid}} + \omega'(\tau - t_{\text{mid}})$
ω_{mid}: filter frequency at t_{mid}
ω': rate of change of the filter frequency

α_i's: parameters of the modulating function. Directly estimated from:
I_a: Arias intensity
D_{5-95} : effective duration of the motion
t_{mid}: time at which 45% of I_a is reached

Modelling of the PEER database accelerograms

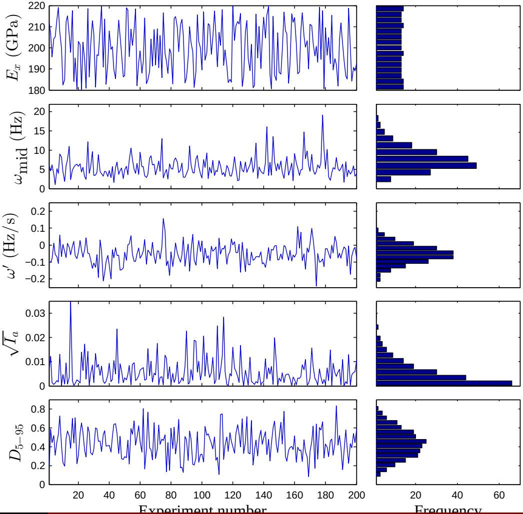
(results from the 1000 accelerograms with the best fit)



Input random vector realizations for the 200 simulations conducted

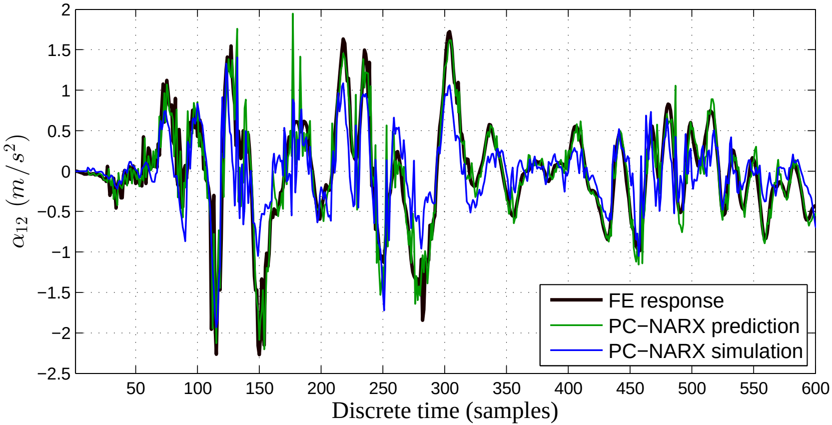
Random input variables

Variable	Distribution	pdf paramete
E (GPa)	Uniform	min = 180 max = 220
ω_{mid} (Hz)	Log-Logistic	$\alpha = 1.64$ $\beta = 0.238$
ω' (Hz/s)	Logistic	$\mu = -0.0516$ $\sigma = 0.0336$
$\sqrt{I_a}$	Exponential	$\mu = 0.00551$
D_{5-95}	Beta	$\alpha = 4.15$ $\beta = 5.15$



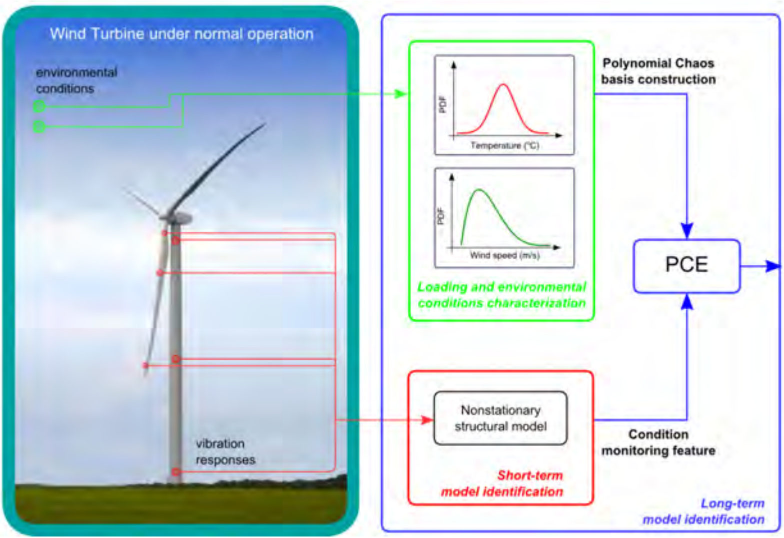
Validation based on a real earthquake ground motion acceleration excitation: FE vs PC-NARX metamodel

(El Centro earthquake time history loading)



Normalized residual sum of squares: prediction 13.15%, simulation 38.10%

Performance Index Extraction for Wind Turbine Systems



Performance Index Extraction for Wind Turbine Systems

Random input variables

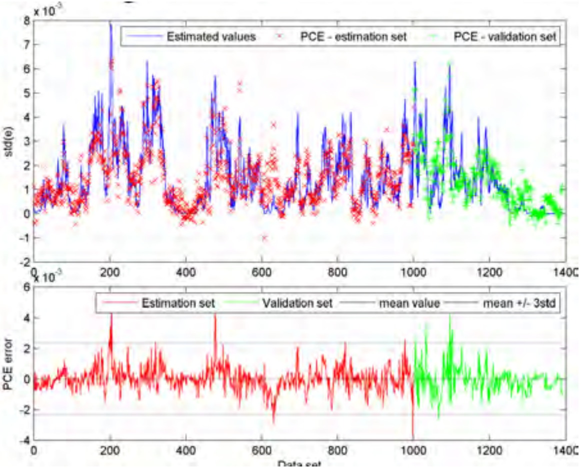
1	Wind direction [deg]
2	Average power [kW]

Output variables

1	Standard deviation of the SP-TARMA model residuals

Polynomial Chaos basis

Legendre polynomials (maximum total order = 7)



References

- M. Spiridonakos and E. Chatzi, "Metamodeling of Structural Systems through Sparse Polynomial Chaos Expansion, Proceedings of International Conference on Noise and Vibration Engineering, September 2012, Leuven, Belgium.
- M. Spiridonakos and E. Chatzi, "Metamodeling of structural systems with parametric uncertainty subject to stochastic excitation, 11th International Conference on Structural Safety & Reliability, June 16-20, 2013, Columbia University, New York, NY
- Spiridonakos, M., and Chatzi, E., "Metamodeling of structural systems with parametric uncertainty subject to stochastic excitation", Special Issue on "Structural Identification and Monitoring with Dynamic Data" in Earthquakes and Structures/ An International Journal for Earthquake Engineering Earthquake Effects on Structures, in press

PC-NARX identification results

Nonlinear regressors:

Initial search space:

$g_i(z[t]) = z_{j_1}^{\ell_1}[t] \cdot z_{j_2}^{\ell_2}[t]$ with $\ell_1, \ell_2 = 0, \dots 3,\ \ell_1 + \ell_2 \leq 3$

$z[t] = [\ y[t-1], \dots, y[t-10], x[t], x[t-1], \dots, x[t-10]\ ]^{\mathsf{T}}$

Finally selected terms:

$y[t-1], \dots, y[t-10], x[t], x[t-1], \dots, x[t-10],$

$y[t-1] \cdot y^2[t-2], \dots, y[t-1] \cdot y^2[t-10],$

$y^2[t-1] \cdot y[t-2], \dots, y^2[t-1] \cdot y[t-10],$

$y^3[t-1], y^3[t-2], y^3[t-3].$

Multi-indices of the selected PC basis functions

	E	ω_{mid}	ω'	I_a	D_{5-95}
$d(1)$	0	0	0	0	0
$d(2)$	1	0	0	0	0
$d(3)$	0	1	0	0	0
$d(4)$	0	0	0	0	1
$d(5)$	1	0	0	1	0
$d(6)$	0	1	0	0	1
$d(7)$	0	2	0	0	0
$d(8)$	1	2	0	0	0
$d(9)$	0	3	0	0	0

Error Levels

PC-NARX based prediction error 4%
PC-NARX based simulation errors 30% (L_2 Norm)